Can Overreaction offer a satisfying modelling of the yield curve that does not require to assume a market price of risk?

Author: Dimitri Tronson

Schools: EDHEC Business School & Université Côte d’Azur

Diploma: Msc in Financial Engineering (EDHEC) & Master of Applied Mathematics (UCA)

Email: dimitri.tronson@gmail.com

Phone: +33610991655

Abstract:

With a simple monetary model and a deviation of inflation from its target following an Ornstein-Uhlenbeck process, we derive first the unbiased yield curve formula assuming the expectation hypothesis holds and we adapt the expression to account for overreaction (biased yield curve). Then, we demonstrate that the biased yield curve is able to reproduce the correlation pattern between predicted and realized changes in yield observed on historical data. After, we analyze the profitability of the unconditional strategy and show that the addition of a negative drift to our model enables to obtain a comparable significantly positive Sharpe ratio as with real data, but it is not enough to conclude on the relevance of dealing with biased yield curve instead of unbiased yield curve. In the last part, we demonstrate that if we consider conditional excess return strategies, then the biased universe enables to get on average a positive Sharpe ratio as with historical data, whereas the unbiased universe leads to a Sharpe ratio that is not significantly different from zero. We deduce that our overreaction model with no risk premium can reproduce the same deviations of the yield curve from the expectation hypothesis as in real data.

Keywords: excess returns, expectation hypothesis, overreaction, yield curve's slope, return predictability

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Introduction

Under modern Fixed Income theory, the yield of a bond is usually thought in terms of expectations of future short rates plus a risk premium and a convexity component. Where does this come from? Historically, it has been observed that interest rates have been significantly cut by the Fed since the 1950s, while the time-average yield curve (Figure 0) has been upward sloping. In addition, the carry strategy consisting in investing in long maturity bonds, by funding with selling short maturity bonds has been profitable on US Treasuries. Hence, a classic explanation consisted in saying that there was a duration risk involved in the excess return strategy, and risk-averse investors required a (positive) market premium to compensate for this duration risk, thus exerting an upward-sloping pressure on the yield curve. However, saying the carry strategy has been profitable on average does not mean it has been profitable at any point in time. On Table 0, we notice that the Sharpe ratio of the carry strategy has been alternatively positive or negative, depending on the market cycle, ranging from 0.82 in recession to 0.01 in expansion and -1.52 in tightening cycles. The subsequent existence of a state-dependent risk premium that changes sign depending on the state of the economy contradicts the idea that investors are always risk averse. In fact, investors generally become risk-seeking in early-stage expansionary cycles by requiring higher duration premium from investing in bonds compared to equity and then turn to be more risk averse at early-stage recession cycle when bonds act as insurance. The problem here is that changing sign risk aversion cannot by itself explain changing sign Sharpe ratios. The profitability of the carry strategy as well as the upward sloping time-average yield curve in a declining rate environment should be explained as an interaction between Fed's monetary policy and how investors react to it (instead of just whether investors become more or less risk averse). The cognitive bias that aims to capture this interaction is overreaction. Under this assumption, investors think that it will be more difficult for the Fed to bring inflation back to its target. If we assume no market price of risk but the existence of an overreaction component, are we able to build a satisfying model of the yield curve? This is the question we will try to solve in the thesis.

	2 years	5 years	10 years
Full Sample	0.20	0.20	0.16
1955-1986	0.04	−0.01	−0.07
1987-2014	0.59	0.56	0.49
Recession	0.82	0.72	0.59
Expansion	0.01	0.06	0.05
1st half Expansion	0.52	0.50	0.45
2nd half Expansion	−0.61	−0.50	−0.48
Tightening Cycles			
1979:Q3-1981:Q2	−1.06	−1.13	−1.23
1993:Q3-1995:Q1	−0.79	−0.86	−0.86
2004:Q2-2006:Q2	−1.52	−0.90	−0.50

Table 0: Sharpe ratio of the carry strategy for several maturities and economic cycles. Data is adapted from Naik, Devarajan, Nowobiski, Page and Pedersen (2016)

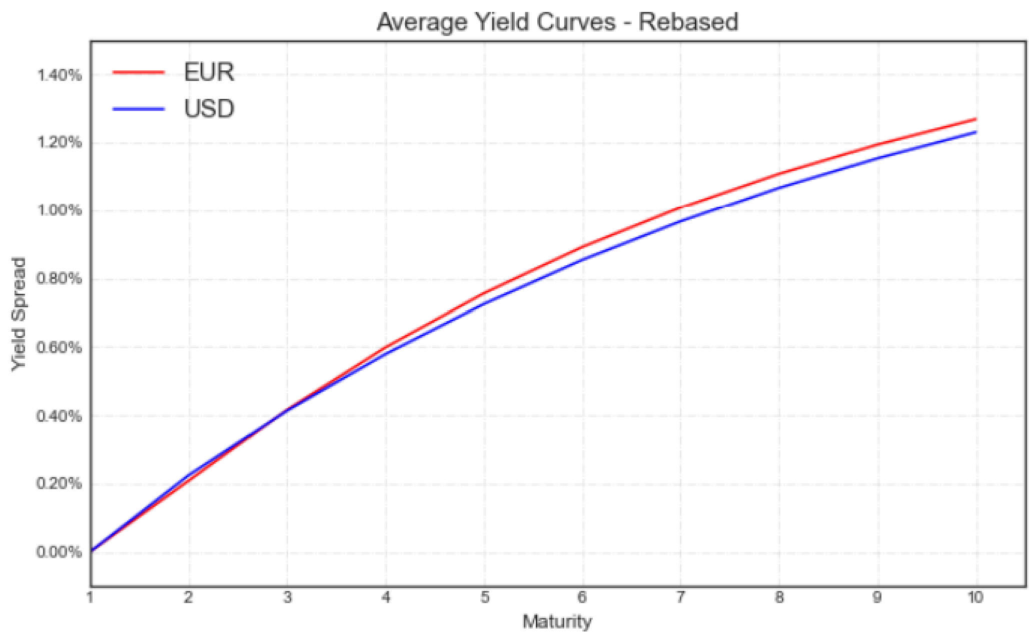


Figure 0: Difference between average yield and 1-year yield of EUR and USD discount bonds, for several maturities.

I) The model that was used to simulate the yield curve

The aim of this part is to simulate the yield curve, under two hypotheses:

- Assumption 1: We are given a simple monetary model, where the deviation of inflation from its Fed's target will follow an Ornstein-Uhlenbeck process.
- Assumption 2: Given short rates and inflation processes, we can derive the yield curve at any maturity, by adding an overreaction component.

We will use the work of Gennaioli and Shleifer (2018) to construct a straightforward monetary model with risk-neutral investors who overreact.

1) Deriving short rate and inflation processes

The aim of a Central Bank is to keep the inflation close to a given target. In the United States and in the Eurozone, this target is generally between 2% and 3%. Every dt year, the Fed observes inflation π_t, how it deviates from its target θ. Then, it rises or decreases the short rate by an amount dr_t. Eventually, this change in dr_t, combined with a stochastic component, will contribute to move inflation by an amount $d\pi_t$. We assume that investors are risk-neutral, so no risk premium should be added in the short rate process.

We call ε_t, the deviation of inflation from its target at time t:

i) $\varepsilon_t = \pi_t - \theta$

The change in the short rate is completely determined by ε_t and is assumed to be proportional to ε_t dt, with a constant A>0 corresponding to how aggressive the Fed is in trying to bring back inflation to its target θ.

ii) $drt = A\ \varepsilon_t\ dt$

It is indeed logic to consider that if the Fed meets less often (dt large), then it will need to make bigger adjustments in the short rate.

Then, inflation will move following a diffusion process:

iii) $d\pi_t = -B\ dr_t + \sigma\ dz_t$

$-B\ dr_t$ indicates how the monetary policy on the short rate is able to affect the inflation process. If inflation is above its target ($\varepsilon_t > 0$), then the Fed is going to hike rates ($A\ \varepsilon_t\ dt > 0$), which will contribute in lowering inflation ($-B\ dr_t < 0$). Conversely, if we are in a recessionary environment with inflation below its target, then the Fed will choose to cut rates so that inflation rises up to its target level.

Finally, from i) we get $d\varepsilon_t = d\pi_t$ and deduce in iii) that $d\varepsilon_t = -B\ dr_t + \sigma\ dz_t$, then, replacing dr_t by its expression in ii) we deduce the following process:

iv) $d\varepsilon_t = -AB\ \varepsilon_t\ dt + \sigma\ dz_t$

We take kappa = AB and can rewrite iv) as following:

v) $d\varepsilon_t = -\text{kappa}\ \varepsilon_t\ dt + \sigma\ dz_t$

We observe a simple stochastic gradient descent (Ornstein-Uhlenbeck process), where the potential is given by:

vi) $V(X) = 0.5 \text{ (kappa) } X^2$

This potential pushes X to the minimizer of V which is 0. Without the stochastic component (ordinary differential equation), $(\varepsilon_t)_t$ would converge in long time to 0. As $\sigma > 0$, we can prove that:

(ε_t) converges in Law to $N(0, \sigma^2/(2 \text{ kappa}))$.

2) Deriving an explicit formula for the yield curve

We would like to get an explicit formula for y_t^T (the yield at time t of a discount bound of maturity T).

step 1:

Starting from t, the solution of v) is:

For any s > t:

$\mathcal{E}_s = \exp(-\text{kappa } (s-t)) (\mathcal{E}_t + \sigma \int_t^s \exp(kappa * u)\, dBu)$, where $(B_t)_t$ is a Brownian motion.

As the function defined by f (u) = exp(kappa*u) is deterministic, and as the Brownian motion has zero expected value, we deduce that:

For any s > t: $E_t (\mathcal{E}_s) = \exp(-\text{kappa } (s-t)) E_t (\mathcal{E}_t)$

As $\mathcal{E}_t$ is measurable with respect to F_t (where $(F_t)_t$ is the natural filtration), then:

$E_t (\mathcal{E}_t) = \mathcal{E}_t$

Finally, we get: $E_t (\mathcal{E}_s) = \exp(-\text{kappa } (s-t)) * \mathcal{E}_t$

step 2:

If we integrate equation ii) between t and s, we get:

$r_s - r_t = A\int_t^s \mathcal{E}(u)\, du$

Now we take the expectation at time t and use the previous result of step 1. We find:

$E_t(r_s) = r_t + A\, \mathcal{E}_t \int_t^s \exp\left(-kappa * (u - t)\right) du$

$$\boxed{\text{Hence, } E_t(r_s) = r_t + A\, \mathcal{E}_t\ \frac{1-\exp\left(-kappa*(s-t)\right)}{kappa}}$$

step 3:

As we assume there is no risk premium, and as we neglect convexity, we have according to the expectation hypothesis:

$y_t^T = \frac{1}{T-t} E_t\left(\int_t^T r(s) ds\right)$

So, using the result from step 2 and the Fubini theorem:

$y_t^T = \frac{1}{T-t} \int_t^T \left(r(s) + A\mathcal{E}(t)\ \frac{1-\exp\left(-kappa*(s-t)\right)}{kappa}\right) ds$

Hence, we get the final result:

$$\textbf{(*)}\ \mathbf{y_t^T = r_t + \frac{A\mathcal{E}(t)}{kappa}\left(1- \frac{1-\exp\left(-kappa*(T-t)\right)}{kappa*(T-t)}\right)}$$

This result holds in the unbiased universe only. If we want to work in the biased universe, we need to introduce a new constant: kappa_bias that is different from kappa. Then, two constants will be required to simulate the biased yield curve:

- kappa for simulating the short rate process $(r_t)_t$ and $(\mathcal{E}_t)_t$
- kappa_bias when we have simulated the previous $(r_t)_t$ and $(\mathcal{E}_t)_t$ processes and we would like to simulate the $(y_t^T)_t$ process.

We get the following equation under overreaction hypothesis:

$$\textbf{(**)}\ \mathbf{y_t^T = r_t + \frac{A\mathcal{E}(t)}{kappa\_bias}\left(1- \frac{1-\exp\left(-kappa\_bias*(T-t)\right)}{kappa\_bias*(T-t)}\right)}$$

Here, kappa_bias = A * B_bias where B_bias < B. It means biased investors behave as if the inflation process reacted less to monetary changes in the short rate than in the unbiased universe. It is equivalent to say that biased investors overestimate the difficulty of the Central Bank to bring back inflation to its target.

II) Application: Simulation of the yield curve on Python

1) Algorithm

```
A = 1
B_bias = 1
B=1.6
kappa_bias = A * B_bias
kappa=A*B
infl_0 = 0.035
theta_infl = 0.025
mat_max = 10
dt = 1/12
sigma_infl = 0.02
r_0 = 0.02
num_fut = 10000  #number of steps in the future
infl_fut = np.zeros(num_fut)
fac = np.sqrt(dt)
r=r_0
```

```
def simu_y(B_bias=1,sigma_infl=0.02,theta_infl=0.025):
    kappa_bias=A*B_bias
    simul_yield_curve=np.zeros((num_fut,mat_max))
    simul_yield_curve_bias=np.zeros((num_fut,mat_max))
    simul_short_rate=np.zeros(num_fut)
    for i in range(num_fut):
        if i==0:
            r=r_0
            infl_fut[i]=infl_0
        if i!=0:
            r=r+A*dt*(infl_fut[i-1]-theta_infl)
            infl_fut[i]=infl_fut[i-1]-kappa*(infl_fut[i-1]-theta_infl)*dt +np.sqrt(dt)*sigma_infl*np.random.randn()
        for j in range(mat_max):
            term_bias=1-(1-np.exp(-kappa_bias*(j+1)))/(kappa_bias*(j+1))
            term=1-(1-np.exp(-kappa*(j+1)))/(kappa*(j+1))
            simul_yield_curve_bias[i][j]=r+(infl_fut[i]-theta_infl)*term_bias/B_bias
            simul_yield_curve[i][j]=r+(infl_fut[i]-theta_infl)*term/B
    return simul_yield_curve, simul_yield_curve_bias
```

```
L_mat=[i+1 for i in range(mat_max)]
simu_yield, simu_yield_bias=simu_y(B_bias=1,sigma_infl=0.01,theta_infl=0.025)
plt.figure(figsize=(8,4))
plt.title('yield curve at time 0', size = 'x-large')
plt.plot(L_mat, simu_yield[0,:],color='brown', label='Unbiased yield curve: B=1.6')
plt.plot(L_mat, simu_yield_bias[0,:],color='blue', label='Biased yield curve: B_bias=1')
plt.xlabel('Maturity in years')
plt.legend();

```

Explanation:

- In the first block, we initialize the variables and parameters. kappa_bias and kappa are equal to 1 and 1.6. The inflation today is 3.5%, and the long-term reversion target of the Fed is 2.5%. The Fed can take decisions every month (dt=1/12) and the initial short rate is 2.0%. The size of the period to simulate is 10000/12=834 years and the maximum maturity for any yield curve will be 10 years.
- In the second block, we define the function simu_y that returns two arrays. The first one contains all the unbiased yield curves from time 0 to time 10,000. The second one contains all the biased yield curves from time 0 to time 10,000. At any time t, (corresponding to index i), we calculate r_t (equation ii), ε_t (equation v). Then, for any maturity T (corresponding to index j), we deduce y_t^T in the unbiased and biased cases (equations * and **).
- In the third block we plot the unbiased and biased yield curves at time 0, with the setting B_bias = 1. And specify a volatility of inflation of 0.02 per year.

2) Results

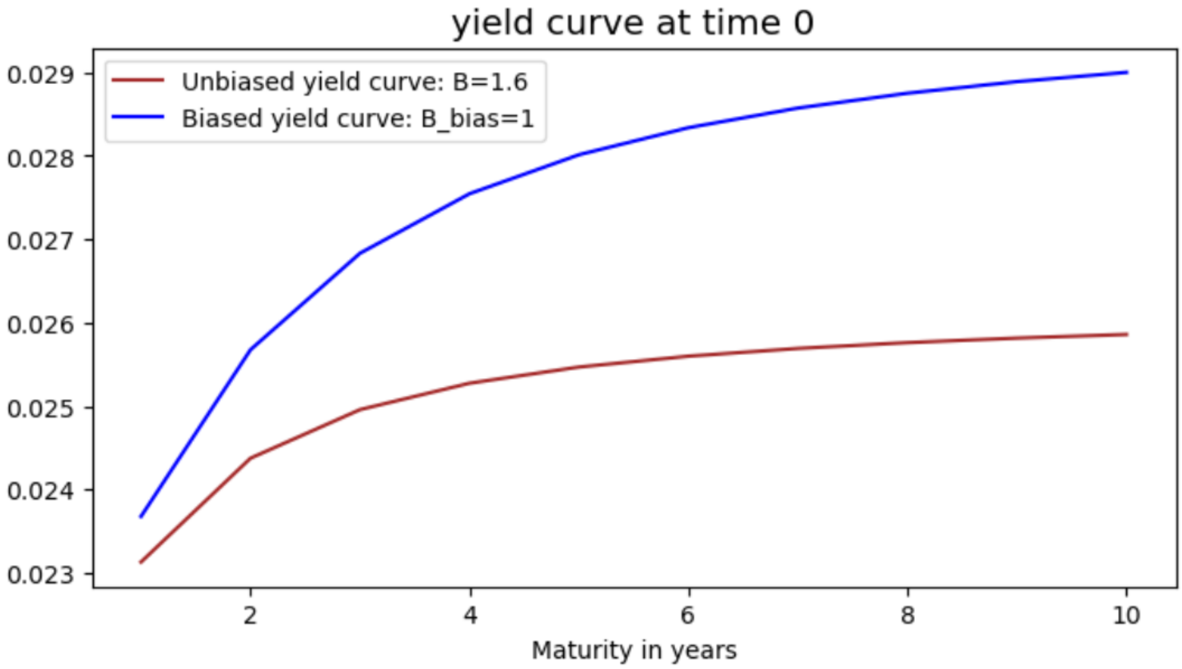


Figure 1: Representation of the unbiased and biased yield curves at t=0

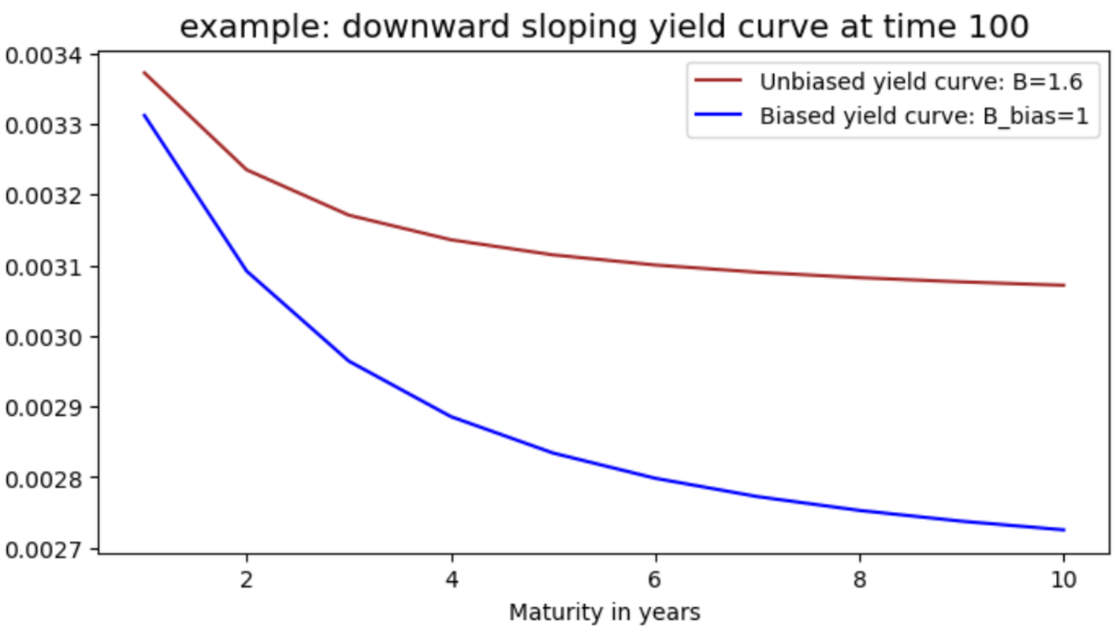


Figure 2: Representation of a downward sloping yield curve at time 100, for a given path.

We can see that the biased yield curve amplifies the slope of the unbiased yield curve in absolute value: if the unbiased yield curve is upward sloping, then the biased yield curve will be even more upward sloping. If the unbiased yield curve is downward sloping, then the biased yield curve will be even more downward sloping.

3) Observation of the inflation and short rate processes

a) Distribution of $(\mathcal{E}_t)_t$ for t large (t=10000)

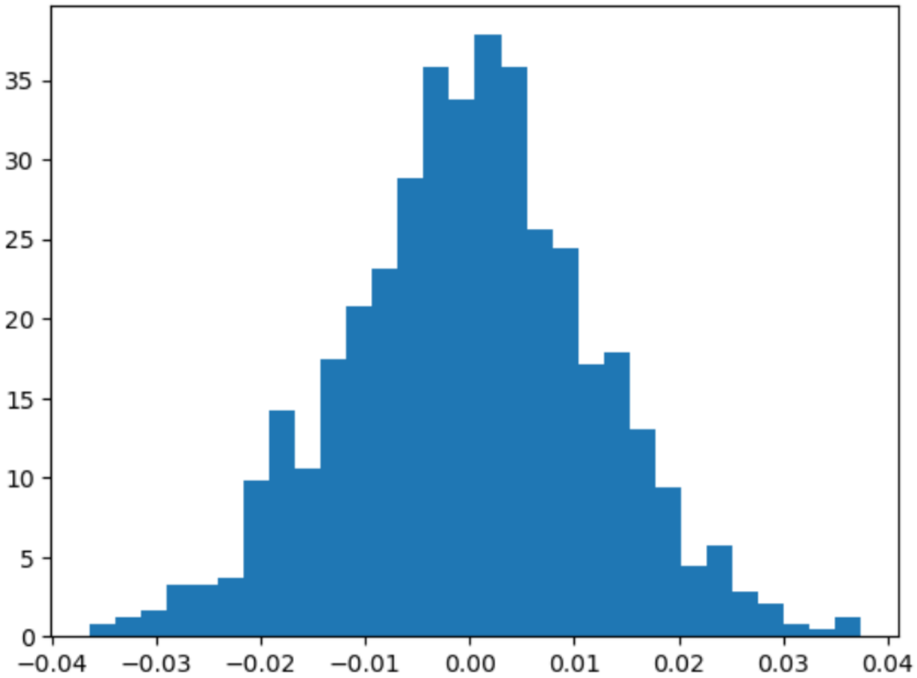


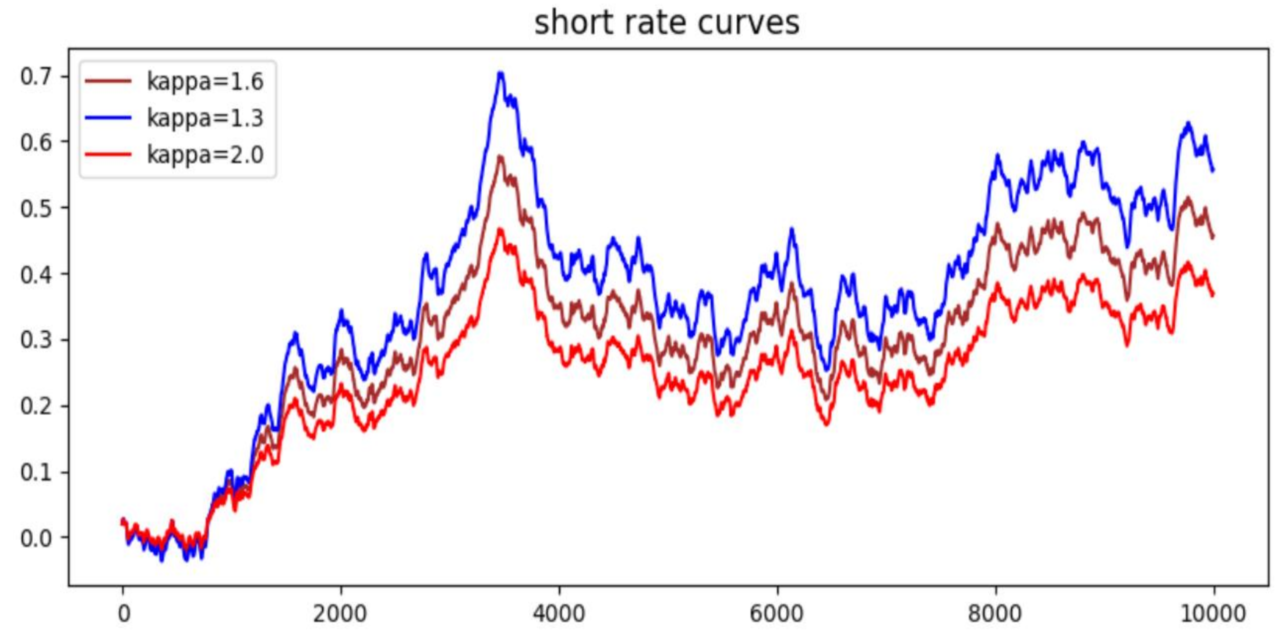
Figure 3: Distribution of 200 paths of $(\mathcal{E}_t)_t$ for t=10000 (we look at terminal value)

When we generate 200 paths of $(\mathcal{E}_t)_t$ processes for a given t (large), the results behave like a normal distribution, centered in 0, with standard deviation tending to s= sqrt(σ^2/(2 kappa)), when t is very large. For σ=0.02 and kappa=1.6, s is close to 0.01. So with a probability of 68%, the inflation deviates from its target by less than 1%.

All the results lie in the range (-0.04, 0.04), but with a very large number of simulations, the probability of staying in the range (-0.03, 0.03) is 99.7%. As $\varepsilon_t = \pi_t - \theta$ (from i), with $\theta = 0.025$, this gives us that all the possible inflation values when t is large lie between -0.5% and 5.5%, but on average inflation is in the long run close to its target of 2.5% since $(\mathcal{E}_t)_t$ has zero expectation.

b) Short rate process

We want to study the effect kappa has on the short rate. For this, we decided to consider three values of kappa: 1.3, 1.6 and 2.0. The algorithm is very similar to the previous one with the simu_y function. The only difference is that we generate three different short rate processes and three different inflation processes (with kappa1, kappa2, kappa3), and we carefully apply the same gaussian variable at each step for these processes. Below are examples of paths we can observe.



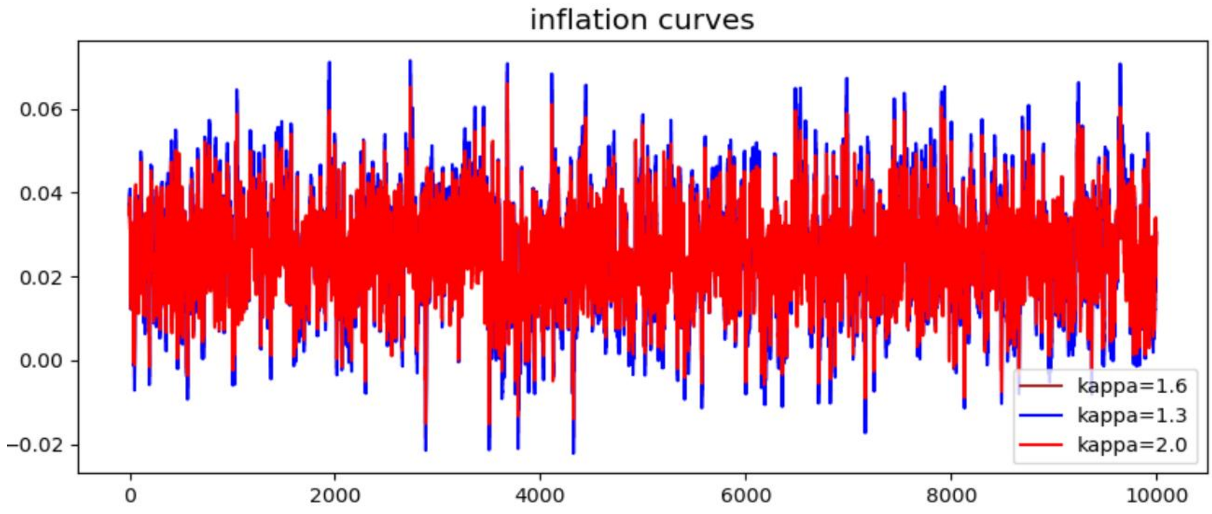


Figure 4: Short rate and inflation curves for one path with three different values of kappa: 1.3 (blue curve), 1.6 (brown curve), 2.0 (red curve)

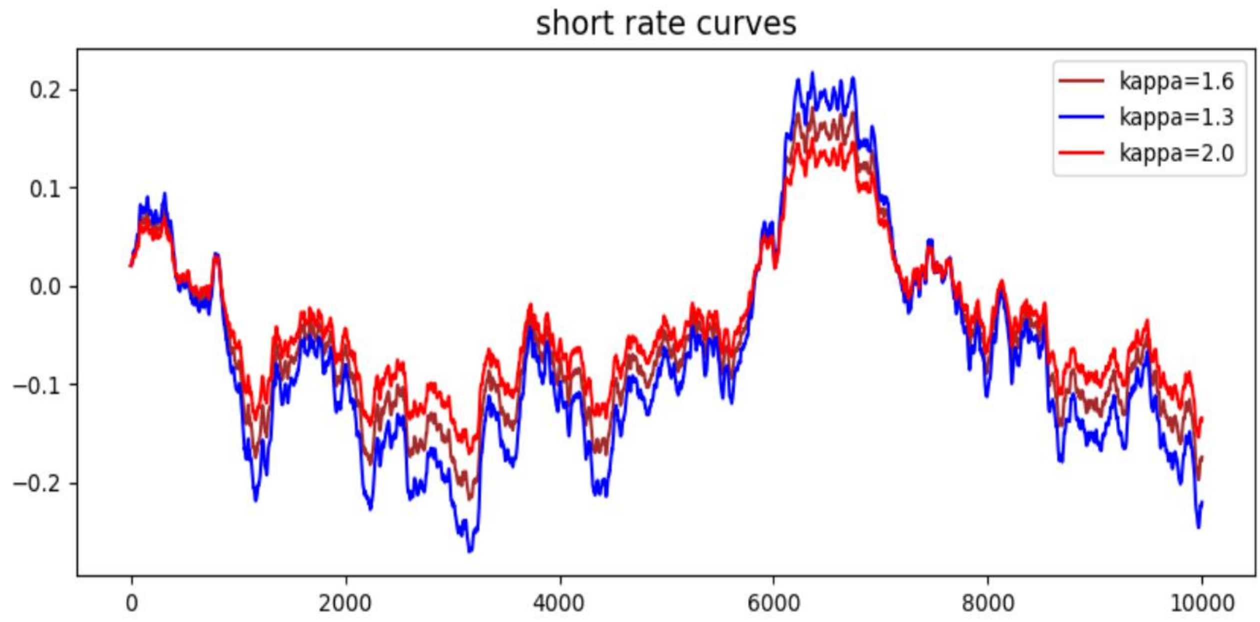


Figure 4: Other path example of short rate curves for three values of kappa

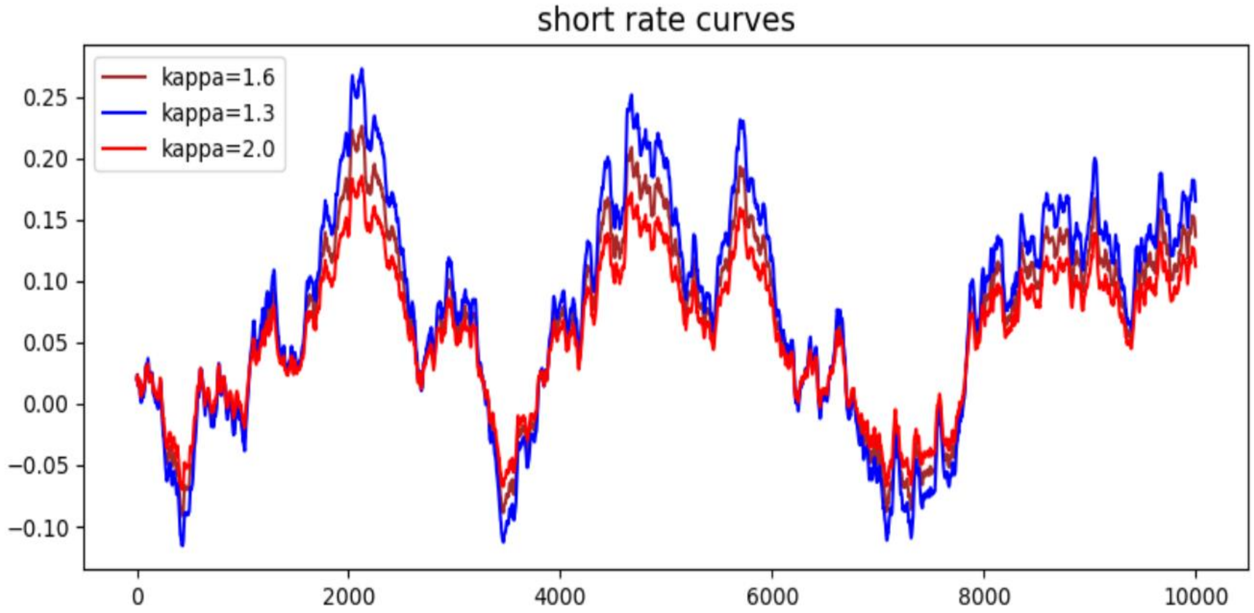


Figure 5: Other path example of short rate curves for three values of kappa

With the three above examples, we notice that for negative values of the short rate, the red curve (kappa = 2) is above the blue and the brown curves, whereas for positive values of the short rate, the red curve is below the blue and the brown curve. Hence, it can be deduced that a high kappa has the effect of reducing the magnitude of the short rate process. This same observation can be applied to the inflation process: the oscillations are higher for low values of kappa (kappa = 1.3) and lower for larger values of kappa (kappa = 2).

III) Analysis of the correlations between predicted and realized changes in yield

1) Methodology

In this section, we intend to check whether the deviations from the expectation hypothesis observed with real data, are also reproduced in our biased universe with simulated data. The first test we are going to perform is whether the 1-period (n-1) maturity forward yield that is set at time t gives an unbiased estimator of the (n-1) maturity future yield that will prevail at time t+1.

We have:

vii) $\mathbf{Fy_t^{n-1} = -\frac{1}{n-1} \log (P_t^n/P_t^1)}$

where Fy_t^{n-1} is the 1-period, (n-1) maturity forward rate that is set at time t, P_t^n is the price at time t of an n-year maturity discount bond, P_t^1 is the price at time t of a 1-year maturity discount bond. The forward rate defined with equation (vii) is equal to the (n-1) maturity yield that would prevail at time t+1 such that the excess return strategy would have zero profitability.

If the expectation hypothesis holds, the predicted change in yield should be equal to the realized change in yield. Working with historical data, biased simulated data and unbiased simulated data, we are therefore going to build the regression:

(viii) $\mathbf{y_{t+1}^{n-1} - y_t^n = \alpha^n + \beta^n (Fy_t^{n-1} - y_t^n)}$

In the case of an univariate regression: Y = BX + A, we have B = Cov(X,Y) / V(X) where Cov(X,Y) in the covariance of X and Y variables, and V(X) is the variance of X. We can therefore write $B = Corr(X,Y) \frac{\sigma(Y)}{\sigma(X)}$ where Corr(X,Y) is the correlation between X and Y, $\sigma(X)$ and $\sigma(Y)$ are the standard deviations of X and Y.

So instead of analyzing coefficients α^n and β^n for each maturity n (between 1 and 9), we are just going to compute $Corr(X,Y)^n$, the correlation between predicted changes in yields (variable X) and realized changes in yields (variable Y), for maturity n.

In the absence of a risk-premium (risk-neutral investors), if the expectation hypothesis was true, then predicted and realized changes in yields should have a correlation curve that is decreasing with maturities, but which is always positive. We could expect to have a correlation curve that tends to zero when maturity increases, and the maximum is reached for short maturity (1-year in our case). It is logic since in the short run, future yields should be more impacted by their predictions (forward yields) than in the long run.

2) Python Algorithm used for the simulated data

```
def correl(B_bias=1,B=1.6, sigma_infl=0.01,theta_infl=0.025):

    kappa=A*B
    kappa_bias=A*B_bias
    simul_yield_curve=np.zeros((num_fut,mat_max))
    simul_yield_curve_bias=np.zeros((num_fut,mat_max))
    simul_short_rate=np.zeros(num_fut)
    discount_bond=np.zeros((num_fut,mat_max))
    discount_bond_bias=np.zeros((num_fut,mat_max))
    forward_bond=np.zeros((num_fut,mat_max-1))
    forward_yield=np.zeros((num_fut,mat_max-1))
    forward_bond_bias=np.zeros((num_fut,mat_max-1))
    forward_yield_bias=np.zeros((num_fut,mat_max-1))

```

```
    for i in range(num_fut):
        if i==0:
            r=r_0
            infl_fut[i]=infl_0
        if i!=0:
            r=r+A*dt*(infl_fut[i-1]-theta_infl)
            infl_fut[i]=infl_fut[i-1]-kappa*(infl_fut[i-1]-theta_infl)*dt +np.sqrt(dt)*sigma_infl*np.random.randn()
        for j in range(mat_max):
            term_bias=1-(1-np.exp(-kappa_bias*(j+1)))/(kappa_bias*(j+1))
            term=1-(1-np.exp(-kappa*(j+1)))/(kappa*(j+1))
            simul_yield_curve_bias[i][j]=r+(infl_fut[i]-theta_infl)*term_bias/B_bias
            simul_yield_curve[i][j]=r+(infl_fut[i]-theta_infl)*term/B
            discount_bond_bias[i][j]=np.exp(-simul_yield_curve_bias[i][j]*(j+1))
            discount_bond[i][j]=np.exp(-simul_yield_curve[i][j]*(j+1))
            if j>0:
                forward_bond_bias[i][j-1]=discount_bond_bias[i][j]/discount_bond_bias[i][0]
                forward_bond[i][j-1]=discount_bond[i][j]/discount_bond[i][0]
                forward_yield_bias[i][j-1]=-np.log(forward_bond_bias[i][j-1])/j
                forward_yield[i][j-1]=-np.log(forward_bond[i][j-1])/j

```

```
    realized_yield_change_bias=simul_yield_curve_bias[12:,0:mat_max-1]-simul_yield_curve_bias[0:num_fut-12,0:mat_max-1]
    predicted_yield_change_bias=forward_yield_bias[0:num_fut-12,0:mat_max-1]-simul_yield_curve_bias[0:num_fut-12,0:mat_max-
    realized_yield_change=simul_yield_curve[12:,0:mat_max-1]-simul_yield_curve[0:num_fut-12,0:mat_max-1]
    predicted_yield_change=forward_yield[0:num_fut-12,0:mat_max-1]-simul_yield_curve[0:num_fut-12,0:mat_max-1]
    L_cor_bias=np.zeros(9)
    L_cor=np.zeros(9)
    for k in range(9):
        c_bias=pearsonr(realized_yield_change_bias[0:num_fut-12,k],predicted_yield_change_bias[0:num_fut-12,k])
        L_cor_bias[k] = c_bias[0]
        c=pearsonr(realized_yield_change[0:num_fut-12,k],predicted_yield_change[0:num_fut-12,k])
        L_cor[k] = c[0]

    return L_cor,L_cor_bias

```

Explanation: We create a function correl that takes B, B_bias, sigma_infl, theta_infl as arguments and returns two lists. Both lists return the correlations between predicted and realized changes in yields, for maturities going from one year to nine year. The difference is that the first list refers to unbiased data (kappa = kappa_biased), while the second list is obtained in the biased universe (kappa_bias < kappa).

From line 3 to 13, we just create all the arrays we will use later.

From line 15 to 28, the code is almost the same as what we did to simulate the yield curve previously. We just added the calculation of unbiased and biased discount bond prices at time i for maturity j.

From line 29 to 33, we calculate the prices of unbiased and biased 1-period forward bonds and deduce the unbiased and biased 1-period forward yields (at time i, with maturity j-1). The calculation corresponds to equation (vii).

From line 35 to 38, we compute the realized and predicted changes in yields, in the unbiased and biased cases. For a given time t, and a given maturity n, the realized change in yield is the difference between the n-maturity yield at time t and the n-maturity yield at time t-12 (one year before). For the predicted change, we consider the 1-period forward rate (that was set at time t-12) instead of the yield at time t.

From line 41 to 45, we loop on all maturities j (going from 1 to 9) and calculate the corresponding correlations between predicted and realized changes in yield, in the unbiased and biased cases.

3) Calculation of the correlations on historical data

The historical data we have chosen comes from Gurkaynak, Sack, and Wright (2007). It corresponds to the yields of synthetic discount bonds obtained by least-square fit to the prices of nominal and real Treasuries using the popular Nelson and Siegel (1987) model. They cover the period 31-Dec-1971 to 30-Sep-2020.

Starting from monthly data of USD Treasury bills, from 12/31/1971 to 09/30/2020, with maturity ranging from 1 to 10 years, we implemented the same principle as before to build the realized change table and the predicted change table.

	Correl	0.10989	0.035343	-0.01591	-0.05607	-0.08712	-0.11102	-0.12962	-0.1444	-0.15641
	t_stat	**2.644207**	0.845815	-0.38052	-1.34302	**-2.09157**	**-2.67165**	**-3.12648**	**-3.49013**	**-3.78732**
		pred_Ch								
		1 year	2 years	3 years	4 years	5 years	6 years	7 years	8 years	9 years
31-déc-71		0.008741	0.006947	0.005644	0.004681	0.003956	0.003401	0.002967	0.002623	0.002346
31-janv-72		0.013534	0.009555	0.007135	0.005588	0.004548	0.003817	0.003281	0.002874	0.002556
29-févr-72		0.012881	0.009379	0.007147	0.005666	0.004645	0.003913	0.003371	0.002956	0.00263
31-mars-72		0.011015	0.006403	0.004366	0.003286	0.002631	0.002192	0.001879	0.001644	0.001462
30-avr-72		0.012615	0.008396	0.006059	0.004659	0.003757	0.00314	0.002693	0.002357	0.002096
31-mai-72		0.009834	0.007354	0.005708	0.00458	0.003783	0.003202	0.002765	0.002428	0.002163
30-juin-72		0.00864	0.005579	0.003964	0.003027	0.002434	0.002031	0.001742	0.001524	0.001355
31-juil-72		0.012433	0.007479	0.005156	0.003893	0.003119	0.0026	0.002229	0.00195	0.001733
31-août-72		0.007948	0.005348	0.003883	0.002994	0.002418	0.002022	0.001735	0.001518	0.00135
30-sept-72		0.007514	0.004804	0.003398	0.002589	0.00208	0.001736	0.001488	0.001302	0.001158
31-oct-72		0.007964	0.004663	0.003186	0.002399	0.001921	0.001601	0.001372	0.001201	0.001067
30-nov-72		0.007484	0.004801	0.003401	0.002593	0.002084	0.001739	0.001491	0.001305	0.00116
31-déc-72		0.005268	0.003331	0.002344	0.001782	0.001431	0.001194	0.001023	0.000896	0.000796
31-janv-73		0.002821	0.001458	0.000943	0.000697	0.000555	0.000462	0.000396	0.000346	0.000308
28-févr-73		0.003251	0.001411	0.00057	0.000191	2.4E-05	-4.6E-05	-7.2E-05	-7.9E-05	-7.7E-05
31-mars-73		-0.00245	-0.00212	-0.00159	-0.00122	-0.00098	-0.00082	-0.0007	-0.00061	-0.00055
30-avr-73		-0.00059	-0.00077	-0.00057	-0.00044	-0.00035	-0.00029	-0.00025	-0.00022	-0.00019
31-mai-73		-0.00428	-0.00225	-0.0015	-0.00112	-0.0009	-0.00075	-0.00064	-0.00056	-0.0005
30-juin-73		-0.00981	-0.00513	-0.00342	-0.00257	-0.00205	-0.00171	-0.00147	-0.00128	-0.00114
31-juil-73		-0.01256	-0.00776	-0.00524	-0.00393	-0.00314	-0.00262	-0.00225	-0.00197	-0.00175

	-0.00042	-0.07128	-0.11046	-0.1387	-0.15934	-0.1747	-0.18661	-0.19624	-0.20425
	-0.01	-1.70919	-2.65803	-3.34961	**-3.86018**	**-4.24338**	**-4.54278**	**-4.78655**	**-4.99014**
	real_change								
	1 year	2 years	3 years	4 years	5 years	6 years	7 years	8 years	9 years
31-déc-71	0.013028	0.011292	0.00946	0.007877	0.006588	0.005557	0.004733	0.004069	0.003528
31-janv-72	0.018949	0.013593	0.009981	0.007579	0.00594	0.004781	0.003929	0.003282	0.002776
29-févr-72	0.022309	0.017494	0.013787	0.010984	0.008869	0.007258	0.006015	0.00504	0.004262
31-mars-72	0.019697	0.012964	0.009527	0.007605	0.006418	0.005621	0.005051	0.004623	0.00429
30-avr-72	0.021856	0.015252	0.011342	0.008996	0.007491	0.006462	0.00572	0.005161	0.004726
31-mai-72	0.024652	0.017594	0.013545	0.010917	0.0091	0.00779	0.006812	0.006061	0.005468
30-juin-72	0.025119	0.015893	0.011826	0.009608	0.008235	0.007309	0.006645	0.006147	0.005759
31-juil-72	0.039173	0.026675	0.020685	0.017512	0.015585	0.014297	0.013377	0.012686	0.012149
31-août-72	0.029681	0.018971	0.013376	0.010364	0.00868	0.007695	0.007087	0.006688	0.006412
30-sept-72	0.018274	0.010482	0.006555	0.004635	0.003784	0.003507	0.003532	0.003708	0.003951
31-oct-72	0.01559	0.009091	0.00666	0.005483	0.004794	0.00434	0.004016	0.003773	0.003584
30-nov-72	0.018599	0.01116	0.007973	0.006229	0.005149	0.004421	0.003899	0.003507	0.003202
31-déc-72	0.015454	0.00868	0.006235	0.005321	0.004959	0.0048	0.00472	0.004673	0.004641
31-janv-73	0.008547	0.004557	0.003899	0.004056	0.00435	0.00462	0.004837	0.005008	0.005144
28-févr-73	0.005536	0.002179	0.001884	0.002133	0.002488	0.002833	0.003136	0.003392	0.003605
31-mars-73	0.011668	0.008598	0.008003	0.007828	0.007748	0.007699	0.007665	0.007639	0.007619
30-avr-73	0.020493	0.016119	0.014199	0.013137	0.012476	0.01203	0.01171	0.01147	0.011283
31-mai-73	0.014197	0.012529	0.011605	0.011085	0.010766	0.010552	0.010399	0.010284	0.010195
30-juin-73	0.009572	0.012036	0.012134	0.01183	0.011496	0.011205	0.010965	0.010771	0.010613
31-juil-73	-0.00047	0.004587	0.006482	0.006929	0.00678	0.00635	0.005773	0.005117	0.004417

Figure 6: Calculation of the correlations on historical data

From these tables, we are able to calculate the corresponding correlations in Excel and perform a test-statistic.

This test-statistic enables us to assess the significance of the correlations calculated previously.

By definition, for this test, we have:

(ix) $\mathbf{t = r\sqrt{\dfrac{N-2}{1-r^{\wedge}2}}}$ **, where r in the correlation coefficient**

We consider the test:

H0: “Predicted change and realized change variables are not correlated”, and we take a 95% confidence interval. Using equation (ix), the t-stats are the following:

Correl	0.10989	0.035343	-0.01591	-0.05607	-0.08712	-0.11102	-0.12962	-0.1444	-0.15641
t_stat	**2.644207**	0.845815	-0.38052	-1.34302	**-2.09157**	**-2.67165**	**-3.12648**	**-3.49013**	**-3.78732**

- For 1 year maturity, the t_stat is 2.644207, which is greater than 1.96, so we reject the null with a 95% confidence level.
- For 2,3 and 4 year maturities, the t_stats are 0.85, -0.38, -1.34 which are less than 1.96 in absolute value. So we do not reject the null hypothesis.
- For maturities greater than 5 years, all the t_stats are greater than 1.96 in absolute value, so we reject the null with a 95% confidence level.

Hence, if for short to medium maturities, the results are quite in accordance with the expectation hypothesis (correlation is significantly positive for small maturities, and then decreases to zero and become insignificant when we move to longer maturities), for long maturities (greater than 5 years), the results deviate from the expectation hypothesis. This is caused by correlations becoming significantly negative as well as even more negative when the maturity increases.

4) Results of correlations in the unbiased & biased universe and comparison with historical data

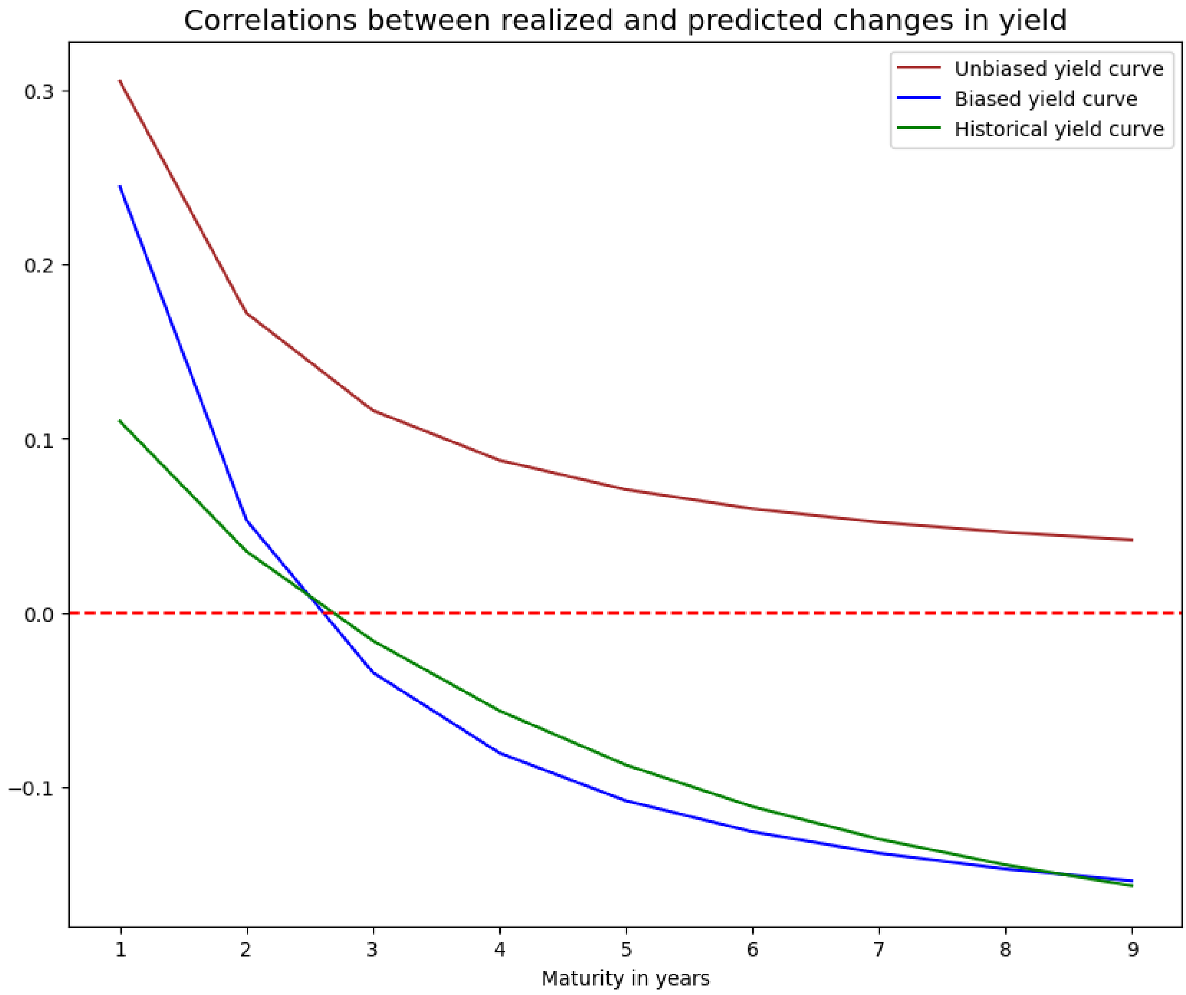


Figure 7: Correlation curves for historical data, unbiased simulated data and biased simulated data, for maturity going from 1 year to 9 years.

The results validate the patterns we expected before:

- The unbiased simulated data is in accordance with the expectation hypothesis.
- The biased simulated data is able to reproduce the same deviation pattern from the expectation hypothesis as the historical data, with positive correlation at the short end, negative and decreasing correlations at the long end. In addition, the curves seem to be very close to each other, and even cross, which is a good sign! It may also suggest that the choice of parameters that was used to simulate the biased yield curves

(kappa = 1.6, kappa_bias = 1) is acceptable for explaining the real yield curves' behaviors.

Interpretation: Positive correlations at the short end is plausible since future yields will be more affected by expectations. Negative correlations can indicate that future yields will correct for overreaction: since investors overreact in the biased universe: a positive predicted change in yield will have a tendency to be associated with a negative realized change in yield and vice versa. Also, the lower kappa_bias compared to kappa (the more investors overreact), the more negative will be the correlations for long maturities i.e. the more aggressive will be the market correction.

If a clear correlation similarity can be identified between the biased model and historical data, what can we say about the profitability of the carry strategy? Do we observe analogous Sharpe ratios? This is the question we are going to tackle in part IV.

IV) Comparison of biased and unbiased models with historical data, in terms of profitability of the excess return strategy

1) The unconditional excess return strategy

We recall that in terms of yields, the excess return generated by investing in a bond of maturity n, funded by the borrow of a bond of maturity 1 year will be approximately in one year:

$$\textbf{(x)}\quad \mathbf{xret_t^n = ny_t^n - (n-1)y_{t+1}^{n-1} - y_t^1}$$

At each time t, we calculate $xret_t^n$ given by equation (x), and then we take the Sharpe ratio (mean divided by standard deviation of all $(xret_t^n)_t$ over the full sample (10,000 simulations in our case). Finally, we take the mean of the sharpe ratios for all maturities and we observe the following results:

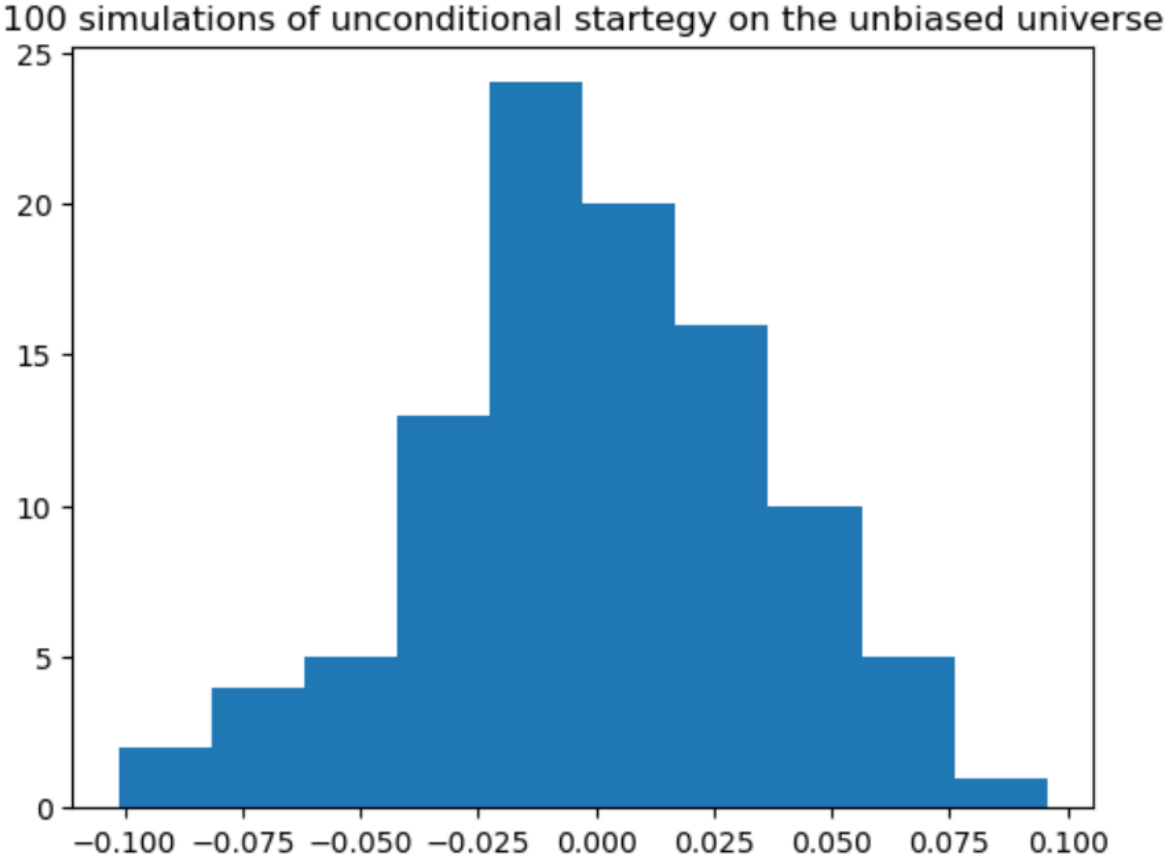


Figure 8: Histogram of the distribution of unconditional strategy on the unbiased universe

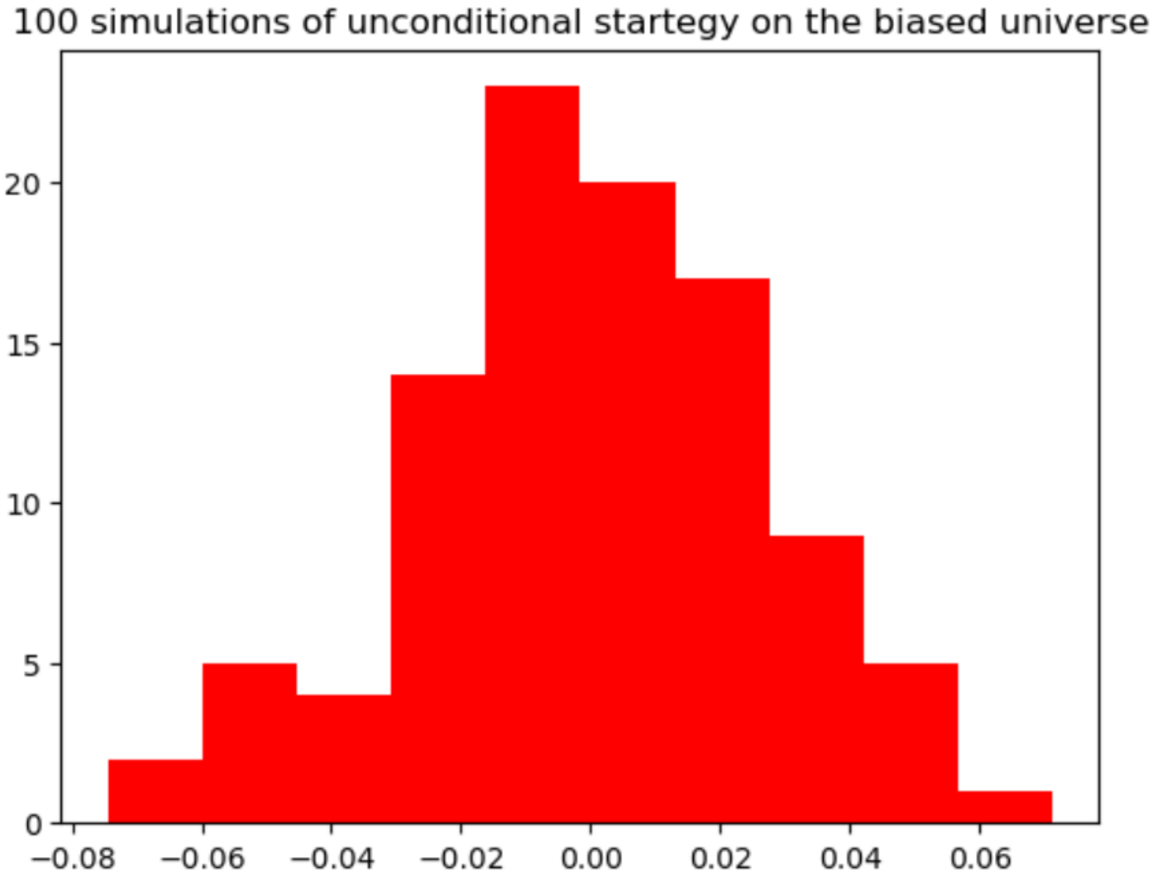


Figure 9: Histogram of the distribution of unconditional strategy on biased universe

So for the biased and unbiased universe, the unconditional strategy follows a Gaussian distribution, centered at zero, with a standard deviation that is not significantly different from zero. We can therefore say that on average, the unconditional strategy has zero expected return. This can be explained by two different reasons:

- In the unbiased universe, the expectation hypothesis holds, so forward rates come true and the excess return strategy yields zero expected return.
- In the biased universe, the expectation hypothesis is not satisfied. However, the profits made when the yield curve is upward sloping are offset by the loss made when the yield curve is downward sloping. Indeed, as there is zero drift about future interest

rates, the probability of having rising or decreasing interest rates is the same at each time t (we do not take a directional bet on interest rates). We understand here why the unconditional excess return strategy leads on average to a Sharpe ratio of zero.

What do we observe on real data?

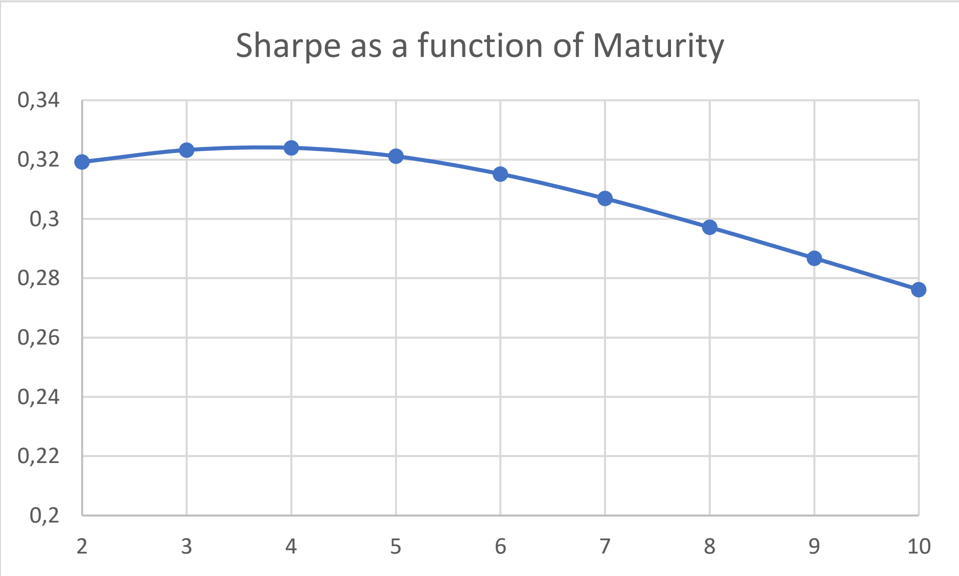


Figure 10: Sharpe ratio as a function of period of investment for the unconditional strategy, on full sample data (from 31-dec-71 to 30-sept-2020)

On real data, we observe significantly positive Sharpe ratios on the full sample, for the unconditional strategy. There is no reason to think that it is necessarily due to the existence of a risk premium. We can interpret this discrepancy between real and simulated data by the existence of an additional negative drift in interest rates on historical data, that is equal to zero in the simulated data. A negative drift is clearly profitable for the excess return strategy since it will enable on average to borrow each year short-term maturity bonds at a lower rate, while keeping the yields of long-term bonds unchanged during the period of investment.

Let's see how our simulated data reacts to the addition of drift comparable to what we observe on real data.

If we assume a drift of -2.5 bps per month, we can obtain results that are quite close to the Sharpe ratio curve above.

We would like to quantify the impact the drift has on the Sharpe ratio. To do this, we will focus only on a 2-year investment horizon. As the Sharpe ratio values are simulation dependents, for each drift, we will simulate 100 paths of yields curves, and take the average of the subsequent 100 Sharpe ratios.

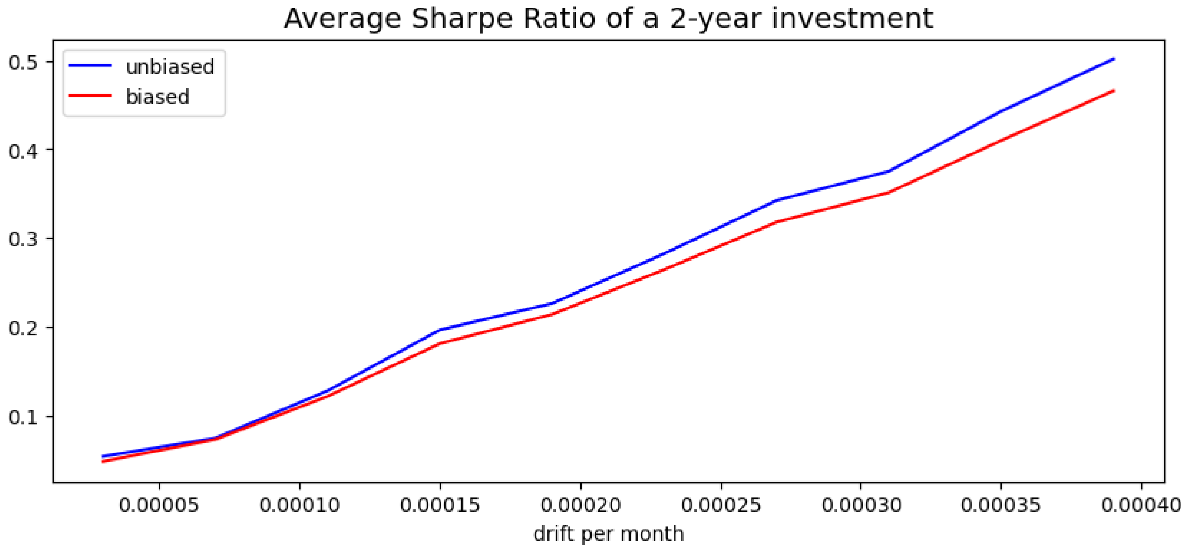


Figure 11: Average Sharpe Ratios of the excess return strategy in the unbiased and biased universe, as a function of the additional negative drift (in the short rate) per month.

We notice that the drift has a significant impact on the average Sharpe ratio: for a drift close to zero, we obtain a Sharpe ratio that is not significantly different from zero, which is in accordance with our previous results of the unconditional strategy. Then, the average Sharpe Ratio is an increasing function of the (negative) drift. For a drift of -2.5 bps per month, we get a Sharpe ratio of about 0.3, which is consistent with what we observe on real data (Sharpe Ratio = 0.32).

Summary: the addition of a drift enables to explain why the Sharpe Ratio is significantly different from zero in the real data, compared to simulated data for both biased and unbiased universe. However, not only are we assuming that rates are constantly declining (and this assumption may not hold in the future, as we currently see with the Fed rates that are close to 5% on May 2023, the highest level since 2006), but by adding this drift, assuming overreaction becomes also completely useless, since as we see in Figure 11, the average Sharpe Ratio increases similarly for both biased and unbiased universe.

2) The conditional excess return strategy

We will call slope for time t, with maturity T > 1, the quantity:

$Slope_t^T = y_t^T - r_t$ (difference between the T-year maturity bond and the short rate at time t)

From equation (**), previously derived, it holds in the biased universe that:

$$(**)\ y_t^T - r_t = \frac{A\mathcal{E}(t)}{kappa\_bias}\left(1 - \frac{1 - \exp(-kappa\_bias * (T-t))}{kappa\_bias * (T-t)}\right)$$

Similarly, from equation (*), we have in the unbiased universe:

$$(*)\ y_t^T - r_t = + \frac{A\mathcal{E}(t)}{kappa}\left(1 - \frac{1 - \exp(-kappa * (T-t))}{kappa * (T-t)}\right)$$

The sign of the slope is then completely determined by the position of inflation, above or below its target. If inflation is above its target, investors think that the Fed is going to hike rates in the future and the yield curve is upward sloping. Conversely, if inflation is below its target, investors think that the Fed is going to cut interest rates in the future to relaunch the economy, so the yield curve is downward sloping. Hence, the slope enables to capture the economic cycle: expansionary when the slope is positive, in (early-stage) recession when the slope is negative (inverted yield curve).

For what follows, we will take the 1 year maturity yield instead of the short rate, in the formula of the slope. So we will define:

$Slope_t^T = y_t^T - y_t^1$ (differente between T year and 1 year maturity yields at time t)

The conditional excess return strategy consists in:

- Buying T year maturity bonds by borrowing 1 year maturity bonds, when the slope is positive
- Selling T year maturity bonds and lending 1 year maturity bonds, when the slope is negative

We will compare the conditional Sharpe ratios of the biased and unbiased universe. We will consider the unbiased slope (for the unbiased yield curve) and the biased slope (for the biased yield curve).

Let's see how the algorithm works:

```
def cor(B_bias=1,sigma_infl=0.02,theta_infl=0.025):

    num_fut=1000
    kappa_bias=A*B_bias
    simul_yield_curve=np.zeros((num_fut,mat_max))
    simul_yield_curve_bias=np.zeros((num_fut,mat_max))
    simul_short_rate=np.zeros(num_fut)
    discount_bond=np.zeros((num_fut,mat_max))
    discount_bond_bias=np.zeros((num_fut,mat_max))

    x_ret=np.zeros((num_fut,mat_max-1))
    x_ret_biased=np.zeros((num_fut,mat_max-1))
    condx_ret=np.zeros((num_fut-12,mat_max-1))
    condx_ret_biased=np.zeros((num_fut-12,mat_max-1))
    slope=np.zeros(num_fut)
    slope_biased=np.zeros(num_fut)
```

```
    for i in range(num_fut):
        if i==0:
            r=r_0
            infl_fut[i]=infl_0
        if i!=0:
            r=r+A*dt*(infl_fut[i-1]-theta_infl)
            infl_fut[i]=infl_fut[i-1]-kappa*(infl_fut[i-1]-theta_infl)*dt +np.sqrt(dt)*sigma_infl*np.random.randn()
        for j in range(mat_max):
            term_bias=1-(1-np.exp(-kappa_bias*(j+1)))/(kappa_bias*(j+1))
            term=1-(1-np.exp(-kappa*(j+1)))/(kappa*(j+1))
            simul_yield_curve_bias[i][j]=r+(infl_fut[i]-theta_infl)*term_bias/B_bias
            simul_yield_curve[i][j]=r+(infl_fut[i]-theta_infl)*term/B
            discount_bond_bias[i][j]=np.exp(-simul_yield_curve_bias[i][j]*(j+1))
            discount_bond[i][j]=np.exp(-simul_yield_curve[i][j]*(j+1))


        slope[i] = simul_yield_curve[i][mat_max-1]  - simul_yield_curve[i][0]
        slope_biased[i] = simul_yield_curve_bias[i][mat_max-1]  - simul_yield_curve_bias[i][0]
```

```
38     for i in range(num_fut-12):
39         for j in range(mat_max-1):
40             x_ret[i][j]=simul_yield_curve[i][j+1]*(j+2)-simul_yield_curve[i+12][j]*(j+1)-simul_yield_curve[i][0]
41             x_ret_biased[i][j]=simul_yield_curve_bias[i][j+1]*(j+2)-simul_yield_curve_bias[i+12][j]*(j+1)-simul_yield_curve_
42             condx_ret[i][j]=x_ret[i][j]*slope[i]
43             condx_ret_biased[i][j]=x_ret_biased[i][j]*slope_biased[i]
44 
45     SR_uncond = np.mean(x_ret)/np.std(x_ret)
46     SR_uncond_bias = np.mean(x_ret_biased)/np.std(x_ret_biased)
47     SR_cond = np.mean(condx_ret)/np.std(condx_ret)
48     SR_cond_bias = np.mean(condx_ret_biased)/np.std(condx_ret_biased)
49 
50     return SR_cond,SR_cond_bias
51 
```

Figure 12: Conditional strategy Algorithm

On this Algorithm in Figure 12, lines 1 to 32 are very similar to what was done previously and consist in building the unbiased and biased yield curves. On lines 34 and 35, we calculate the biased and unbiased slopes at time t (difference between the 10 year and 1 year maturity yields). From line 38 to 48, we make the difference between excess returns and conditional excess returns. While excess returns have been previously defined with equation (x): ((x) $xret_t^n = ny_t^n – (n-1)y_{t+1}^{n-1} – y_t^1$), conditional excess returns are obtained by multiplying the excess returns by the slope, at each time t. If the slope is positive (upward sloping yield curve), then a positive excess return implies a positive conditional excess return, which leads to a gain at time t. Conversely, if the slope is negative (downward sloping yield curve), as we apply a reverse carry strategy, a negative excess return will lead to a positive conditional excess return (and therefore a profit at t). In addition, multiplying the excess return by the slope, (instead of just its sign) also enables to account for the magnitude of the conviction on how strongly we should go carry or reverse carry.

The results are the following:

For the conditional carry strategy, we implemented 300 simulations on the unbiased universe

```
L_sim=[cor(1)[0] for m in range(300)]
sharpe_u=np.mean(L_sim)
sharpe_u
```

```
5.517315721935312e-05
```

```
s_inc=0
for u in range(300):
    if L_sim[u]>=0:
        s_inc+=1
```

```
s_inc/300
```

```
0.5166666666666667
```

Figure 13: results of the conditional strategy on the unbiased universe

We observe on Figure 13, that the average of all the simulated Sharpe ratios of the conditional strategy on the unbiased universe is very close to zero (about 5×10^{-5}). In addition, the probability of observing a positive Sharpe ratio is close to 50% (51.7%). Those results are in accordance with the intuition, and the fact that interest rates have zero drift (contrary to the assumption made on previous part).

For the biased universe, we would like to study how the choice of B_{bias} affect the value of the Sharpe ratio. So, for a range of B_{bias} between 0.8 and 1.6 (1.6 is the value of B so it is an upper bound for B_{bias}) we computed the average Sharpe ratio (based on 100 simulations for each value of B_{bias}). Calculating average values enable to remove the noise in the randomness of Sharpe ratios, and to isolate the effect of B_{bias} on the Sharpe ratios. We got the following curve:

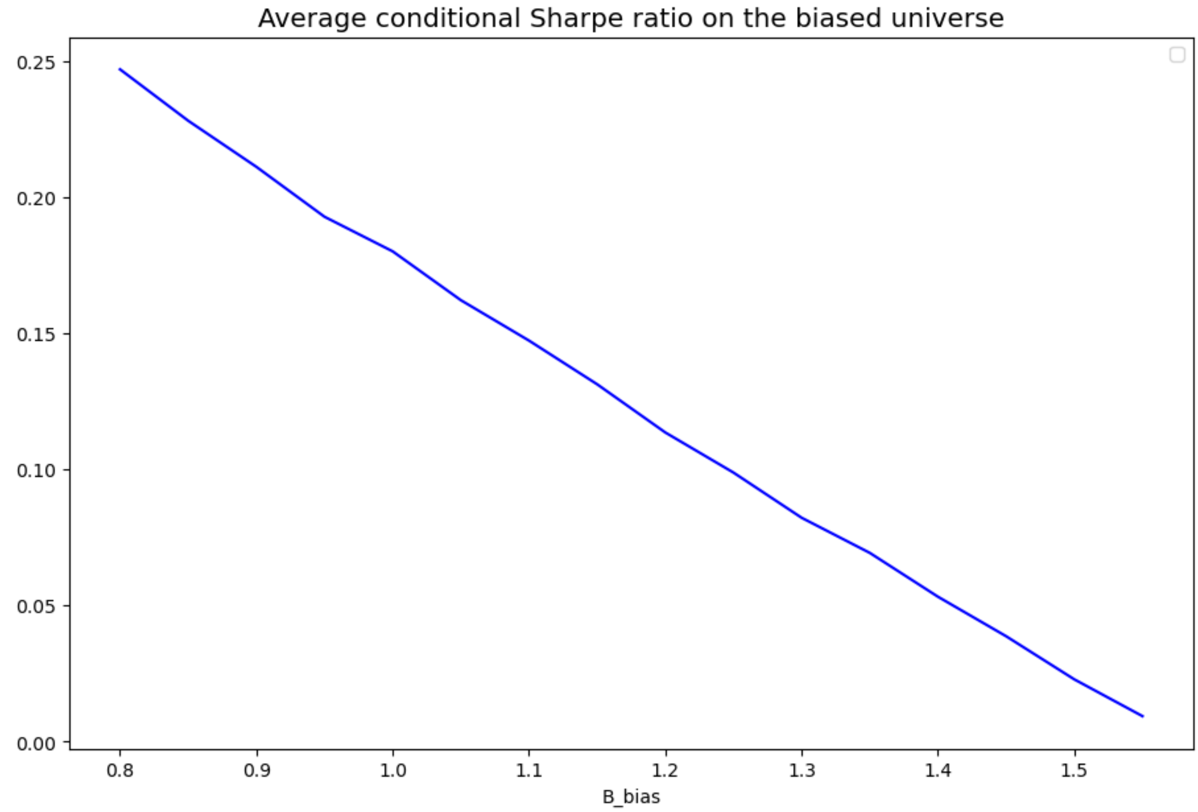


Figure 14: Effect of B_{bias} on the average Sharpe ratio value, by applying the conditional strategy on the biased universe

We observe a satisfying and consistent result. We can observe a (negative) and almost linear relationship between B_{bias} and the average Sharpe ratio. When there is no overreaction (B_{bias} = B = 1.6), the average Sharpe ratio is close to zero and we recover the previous result for the unbiased universe. When there is overreaction, the greater the magnitude of this overreaction (ie the lower the value of B_{bias}) the higher the average Sharpe ratio.

Now, we would like to see how the conditional strategy has historically performed on real data. So we import the history of USD yields for period 1971-12-31/2020-09-30 into a Python Data frame called df_array2 and we get the following result:

```python
num_fut_hist=586
x_ret_hist=np.zeros((num_fut_hist,9))
condx_ret_hist=np.zeros((num_fut_hist-12,9))
slope_hist=np.zeros(num_fut_hist)


for i in range(num_fut_hist):
    slope_hist[i] = df_array2[i][9]  - df_array2[i][0]
for i in range(num_fut_hist-12):
    for j in range(9):
        x_ret_hist[i][j]=df_array2[i][j+1]*(j+2)-df_array2[i+12][j]*(j+1)-df_array2[i][0]
        condx_ret_hist[i][j]=x_ret_hist[i][j]*slope_hist[i]

SR_cond = np.mean(condx_ret_hist)/np.std(condx_ret_hist)
SR_cond
```

```
0.3885241485691096
```

Figure 15: Implementing the conditional strategy on full sample real data

We can clearly see that the conditional strategy has been profitable, with a Sharpe ratio of about 0.39. It is even higher than most of the average Sharpe ratios obtained on Figure 14, when we let B_{bias} vary.

Conclusion:

A large part of Fixed Income Research employs the traditional rational-asset pricing setting, well described in studies from Fama (1989), Stambaug (1988), Fama and French (1986), Dahlquist and Hasseltoft (2016). By incorporating overreaction through an altered reversion speed in our yield curve model, we have shown that representativeness bias could be a stronger assumption than the existence of a market price of risk, the latter raising the problem of changing sign risk premium depending on the business cycle. We have undertaken correlation analysis and conditional excess return strategies and proved we recovered the same deviations from the expectation hypothesis in the biased universe as with historical data. Of course, we do not exclude a coexistence between risk premium and overreaction: the fact the conditional strategy on real data outperforms the conditional strategy in the biased universe (for several reversion speeds) shows that there should be other factors to consider when modelling the yield curve. It could be the subject of another thesis.

Bibliography

- Bulkley, I., R. Harris, and V. Nawosah (2015): “Can behavioral biases explain the rejections of the expectation hypothesis of the term structure of interest rates?,” Journal of Banking and Finance, 58, 179–193.
- Fama, E., and K. French (1986): “Business Conditions and Excepted Returns on Stocks and Bonds,” Journal of Financial Economics, 17, 175–196.
- Fama, E. (1989): “Term Premia and Default Premia in Money Markets,” Journal of Financial Economics, 25, 23–49.
- Gennaioli, N., and A. Shleifer (2018): A Crisis of Beliefs - Investor Psychology and Financial Fragility. Princeton University Press.
- Naik, V., M. Devarajan, A. Nowobiski, S. Page, and N. Pedersen (2016): Factor Investing and Asset Allocation. CFA Institute Research Foundation.
- Stambaug, R. (1988): “The Information in Forward Rates: Implications for Models of the Term Structure,” Journal of Financial Economics, 22, 613–652.

- ECB website to understand the link between the yield curve and the monetary policy: https://www.ecb.europa.eu/press/key/date/2019/html/ecb.sp191125~b0ecc8e6f0.en.html